J. Fluid Mech. (1994), *vol.* 262, *pp.* 157–169

Dynamics of large-amplitude geostrophic flows: the case of 'strong' beta-effect

By E. S. BENILOV
School of Mathematics, University of New South Wales, PO Box 1, Kensington, NSW 2033, Australia

(Received 1 January 1993 and in revised form 19 August 1993)

This paper examines the dynamics of geostrophic flows with large displacement of isopycnal surfaces. The β-effect is assumed strong, i.e. the parameter $(R_d \cot\theta)/R_e$ (where θ is the latitude, R_d is the deformation radius, R_e is the Earth's radius) is of the order of, or greater than, the Rossby number. A system of asymptotic equations is derived, with the help of which the stability of an arbitrary zonal flow with both vertical and horizontal shear is proven. It is demonstrated that the horizontal and vertical spatial variables in the asymptotic system are separable, which yields a 'horizontal' set of evolutionary equations for the amplitudes of the barotropic and baroclinic modes (the vertical profile of the latter is arbitrary).

1. Introduction

Over the past decade there has been considerable interest in large-amplitude geostrophic flows of a stratified fluid, where the variations of isopycnal surfaces are of the order of their average depths. In most cases those were examined within the framework of the two-layer model (e.g. Killworth, Paldor & Stern 1984; Sakai 1989; Paldor & Ghil 1990; Cushman-Roisin, Sutyrin & Tang 1992), which, in particular, demonstrated (Benilov 1992) that the large-amplitude geostrophic dynamics strongly depend on the ratio of the Rossby number *Ro* to the 'β-effect number'

$$\beta = R_d \cot\theta / R_e, \tag{1}$$

where θ is the latitude, R_e is the Earth's radius,

$$R_d = \frac{(gH\,\delta\rho/\rho_0)^{\frac{1}{2}}}{f}, \tag{2}$$

g is the acceleration due to gravity, $\delta\rho/\rho_0$ is the relative density variation, $f = 2\Omega \sin \phi$ is the Coriolis parameter, Ω is the frequency of the Earth's rotation and H is the total depth of the fluid. In the case of *weak* β-effect,

$$\beta \lesssim Ro^{\frac{3}{2}}, \tag{3}$$

large-amplitude flows are unstable with respect to perturbations of wavelength of the order of R_d. In the transitional range,

$$Ro^{\frac{3}{2}} \ll \beta \ll Ro,$$

the spectral margins of the instability shift towards the short-wave region, and in the limit of *strong* β-effect,

$$\beta \gtrsim Ro, \tag{4}$$

the instability disappears completely (see Benilov 1992). It should be noted, however, that the oceanographic relevance of these asymptotic regimes has not yet been addressed.

The 'two-layer' results obtained by Benilov (1992) have been partly generalized for *continuous* stratification by Benilov (1993). It was demonstrated that in the case of weak β-effect (3), the full set of fluid dynamics equations can be reduced to a relatively simple asymptotic system which exhibits exactly the same short-wave instability as its two-layer analogue.

In this paper the case of strong β-effect (4) will be generalized for continuous stratification. The physical relevance of this regime is discussed in §2. An asymptotic system of governing equations will be derived in §3. The stability of a zonal flow with both vertical and horizontal shear is examined within the framework of the equations derived in §4. It is demonstrated (§5) that the horizontal and vertical spatial variables in the asymptotic equations are separable, which yields a 'horizontal' system for the amplitudes of barotropic and baroclinic modes.

2. Are oceanic fronts geostrophic?

Fronts in the ocean are characterized by large displacement of isopycnal surfaces and therefore provide a natural application for the present work. However, they are commonly assumed ageostrophic or, at most, semi-geostrophic. In this section, we shall analyse experimental data on the North Pacific frontal system and point out a number of examples of geostrophic fronts. We shall also demonstrate that both weak- and strong-β-effect regimes are possible in the ocean.

For this purpose we shall use Roden's (1975) observations of the North Pacific frontal system; attention will be focused on the Kuroshio and Oyashio frontal flows and the subarctic and subtropical fronts. At the location where the measurements were made, the subtropical front could be subdivided into two jets (their axes located at latitudes 27° 30′ N and 31° 30′ N, see figure 9 of Roden's paper); the parameters of these jets will be estimated separately. The Oyashio current, in turn, consisted of up to six separate jets – we shall estimate the parameters of the strongest one and the weakest one (located at longitudes 143° 20′ W and 145° 00′ W, respectively – (see figure 6 of Roden's paper).

It is important to understand that any front can be characterized by either the width of the frontal flow, or the width of the corresponding density front (the difference can be strong, as the former is usually noticeably wider). Since we are going to calculate the Rossby number (which does not characterize the density of the flow, but its velocity), we shall estimate the width of the current. This question will be discussed in more detail later in this section.

The parameters of the North Pacific frontal system are represented in table 1. Evidently, all of these frontal currents are geostrophic ($Ro \ll 1$).

It is also worth noting that the condition of geostrophy

$$Ro = \delta u/(fL) \ll 1, \tag{5}$$

where δu is the velocity variation, entails a constraint on the width L of the geostrophic current. First we shall assume that the flow and density variations are localized in the upper (active) layer of effective depth H_1 (this upper layer is not necessarily thin). Then, taking into account that the geostrophic velocity scale is

$$\delta u = g\frac{\delta\rho}{\rho_0}\frac{\delta H_1}{L},$$

	K	O_1	O_2	SA	ST_1	ST_2
L (km)	70	40	55	85	195	75
δu (m s^{-1})	0.5	0.25	0.1	0.3	0.2	0.45
Ro	0.083	0.069	0.020	0.036	0.014	0.088
$\delta\rho/\rho_0 \times 10^4$	6	3	2	5	6	10
β	0.013	0.008	0.007	0.009	0.021	0.032

TABLE 1. Parameters of the North Pacific frontal currents: L is the horizontal spatial scale of the flow, δu is the velocity variation corresponding to L, Ro is the Rossby number, $\delta\rho/\rho_0$ is the relative density variation and the parameter β is given by (1), where the total depth of the fluid was $H = 5500$ m for all currents. K = Kuroshio; O_1 = Oyashio, the strongest jet; O_2 = Oyashio, the weakest jet; SA = the subarctic front; ST_1 = the subtropical front, northern jet; ST_2 = the subtropical front, southern jet.

where δH_1 is the depth variation of the upper layer, we assume that δH_1 is of the order of H_1 (*large*-amplitude flow):

$$\delta H_1 \sim H_1.$$

We obtain

$$(R'_d/L)^2 = Ro \ll 1, \tag{6}$$

where $R'_d = (gH_1/\delta\rho/\rho_0)^{\frac{1}{2}}/f$ is the upper-layer deformation radius. It should be emphasized that condition (6) is not an additional assumption, but follows singularly from the geostrophy condition (5). In order to reconcile (6) with our understanding of oceanic fronts, it should be noted that L is not the width of the density front, but that of the corresponding density-driven current: the difference between the former and the latter is best illustrated by the two-layer model, where the width of the density front is equal to zero, whereas the geostrophic flow can be arbitrarily wide. Finally, as (6) restricts the square of L'_d/L, L does not have to be very large to satisfy the condition $Ro \ll 1$. It should also be made clear that (6) is not valid for small-amplitude geostrophic flows.

Now, in order to distinguish the cases of strong and weak β-effect, we shall calculate β using (1) and (2). Table 1 shows that K, O_1 and SA correspond to the weak-β-effect regime ($\beta \lesssim Ro^{\frac{3}{2}}$), whereas ST_1 corresponds to the strong-β-effect regime. O_2 and ST_2 should be treated as intermediate cases.

Thus, we conclude that: (i) most of the major oceanic fronts are geostrophic; (ii) there are examples of both strong- and weak-β-effect regimes in the real ocean.

The case of weak β-effect and continuous stratification is covered by Benilov (1993); the case of strong β-effect will be considered in the present paper.

3. Basic equations

The equations, which govern a layer of ideal stratified fluid on the β-plane, are

$$\left.\begin{aligned} &u_t + uu_x + vu_y + wu_z + P_x = (1+\beta y)\,v, \\ &v_t + uv_x + vv_y + wv_z + P_y = -(1+\beta y)\,u, \\ &P_z = -\rho, \\ &u_x + v_y + w_z = 0, \\ &\rho_t + u\rho_x + v\rho_y + w\rho_z = 0. \end{aligned}\right\} \tag{7}$$

Here

$$t = \tilde{t}f, \quad x = \frac{\tilde{x}}{R_d}, \quad y = \frac{\tilde{y}}{R_d}, \quad z = \frac{\tilde{z}}{H},$$

$$P = \frac{\tilde{P} - gz}{gH\,\delta\rho/\rho_0}, \quad u = \frac{\tilde{u}}{R_d f}, \quad v = \frac{\tilde{v}}{R_d f}, \quad w = \frac{\tilde{w}}{Hf}, \quad \rho = \frac{\tilde{\rho} - \rho_0}{\delta\rho};$$

where H is the total depth of the fluid, R_d is given by (2) and the dimensional variables (the spatial coordinates $(\tilde{x}, \tilde{y}, \tilde{z})$, the time $\tilde{t}$, the velocity $(\tilde{u}, \tilde{v}, \tilde{w})$, the pressure $\tilde{P}$ and the density $\tilde{\rho}$) are marked with tildes.

Equations (7) will be scaled in terms of $\epsilon = Ro^{\frac{1}{2}}$:

$$\left.\begin{array}{llll} t = \epsilon^{-3}t', & x = \epsilon^{-1}x', & y = \epsilon^{-1}y', & z = z' \\ P = P', & u = \epsilon u', & \mathrm{v} = \epsilon v', & w = \epsilon^{3}w', \\ \rho = \rho', & \beta = \epsilon^{2}\beta'. & & \end{array}\right\} \quad (8)$$

Scaling (8) corresponds to flows with the following dimensional parameters:

horizontal velocity scale $\sim Ro^{\frac{1}{2}} f R_d$,
horizontal spatial scale $\sim Ro^{-\frac{1}{2}} R_d$,
vertical velocity scale $\sim Ro^{\frac{3}{2}} f R_d$,
vertical spatial scale $\sim H$,
displacement of isopycnal surfaces $\sim H$,
slope of isopycnal surfaces $\sim Ro^{\frac{1}{2}} H/R_d$,
timescale $\sim Ro^{-\frac{3}{2}} f^{-1}$,
$\beta \sim Ro$.

Since (8) is very similar to the scaling of the two-layer equations by Benilov (1992), it will not be discussed in more detail.

Substitution of (8) into (7) gives

$$\epsilon^3 u_t + \epsilon^2(uu_x + vu_y) + \epsilon^3 wu_z + P_x = (1+\epsilon\beta y)\, v, \quad (9a)$$

$$\epsilon^3 v_t + \epsilon^2(uv_x + vv_y) + \epsilon^3 wv_z + P_y = -(1+\epsilon\beta y)\, u, \quad (9b)$$

$$P_z = -\rho, \quad (9c)$$

$$u_x + v_y + \epsilon w_z = 0, \quad (9d)$$

$$\epsilon\rho_t + u\rho_x + v\rho_y + \epsilon w\rho_z = 0. \quad (9e)$$

Equations (9) should be supplemented by the no-flow conditions at the rigid boundaries

$$w = 0 \quad \text{at} \quad z = -1, 0. \quad (10)$$

With the help of (9*a*, *b*), the horizontal velocities (u, v) can be expanded into the quasi-geostrophic series

$$\left.\begin{array}{l} v = P_x - \epsilon\beta y P_x + \epsilon^2[(\beta y)^2 P_x - J(P, P_y)] + O(\epsilon^3), \\ u = -P_y + \epsilon\beta y P_y - \epsilon^2[(\beta y)^2 P_y + J(P, P_x)] + O(\epsilon^3); \end{array}\right\} \quad (11)$$

where $J(P, Q) = P_x Q_y - P_y Q_x$ is the Jacobian operator. Substitution of (11) and (9*c*) into (9*d*, *e*) gives

$$w_z = (\beta - 2\epsilon\beta^2 y)\, P_x + \epsilon J(P, \Delta P) + O(\epsilon^2), \quad (12a)$$

$$\epsilon P_{zt} + (1 - \epsilon\beta y)\, J(P, P_z) + \epsilon w P_{zz} = O(\epsilon^2). \quad (12b)$$

Integrating (12*a*) with respect to z over $[-1, 0]$ and taking into account (10), we obtain

$$(\beta - 2\epsilon\beta^2 y) \int_{-1}^{0} P_x \,\mathrm{d}z + \epsilon \int_{-1}^{0} J(P, \Delta P)\,\mathrm{d}z = O(\epsilon^2). \quad (13a)$$

Now we can omit one of the boundary conditions (10) and keep only

$$w = 0 \quad \text{at} \quad z = -1. \tag{13b}$$

Equations (12) and (13) are asymptotically equivalent to the original system (9) and (10). Expanding the solution into an asymptotic series

$$P = P^{(0)} + \epsilon P^{(1)} + \ldots,$$

$$w = w^{(0)} + \epsilon w^{(1)} + \ldots$$

and keeping the zeroth- and first-order terms, we obtain

$$w_z^{(0)} = \beta P_x^{(0)}, \tag{14a}$$

$$w^{(0)} = 0 \quad \text{at} \quad z = -1, \tag{14b}$$

$$J(P^{(0)}, P_z^{(0)}) = 0, \tag{14c}$$

$$\int_{-1}^{0} P_x^{(0)}\,\mathrm{d}z = 0, \tag{14d}$$

$$P_{zt}^{(0)} + J(P^{(0)}, P_z^{(1)}) + J(P^{(1)}, P_z^{(0)}) + w^{(0)} P_{zz}^{(0)} = 0, \tag{14e}$$

$$\beta \int_{-1}^{0} P_x^{(1)}\,\mathrm{d}z + \int_{-1}^{0} J(P^{(0)}, \Delta P^{(0)})\,\mathrm{d}z = 0 \tag{14f}$$

(the equation and boundary condition for $w^{(1)}$ have been omitted). In principle, now we should solve (14*a–d*) with respect to $P^{(0)}$ and then substitute the solution into (14*e,f*). The latter should be treated as a boundary value problem for $P^{(1)}$, and the condition of its solvability determines the evolutionary equation for $P^{(0)}$. Unfortunately, this standard scheme does not work in our case, as the zeroth-order equations (14*a–d*) cannot be solved in general form. Accordingly, we shall leave (14) as it is and, in each particular case, treat it as a *simultaneous* system for $P^{(0)}$ and $P^{(1)}$. This procedure will be illustrated in §§4 and 5.

It should also be noted that the solution to system (14) is not unique with respect to the first correction $P^{(1)}$. In particular, (10) are invariant with respect to the transformation

$$P^{(1)} \rightarrow P^{(1)} + \mathrm{const}_1\, P_x^{(0)} + \mathrm{const}_2\, P_y^{(0)},$$

which corresponds to the infinitesimal spatial shift of $P^{(0)}$:

$$P^{(0)}(t, x, y, z) \rightarrow P^{(0)}(t, x + \epsilon\,\mathrm{const}_1, y + \epsilon\,\mathrm{const}_2, z).$$

In order to fix $P^{(1)}$, we should consider the next order of the perturbation expansion; however, we shall not do this as all other unknowns in system (11) are fixed independently of $P^{(1)}$.

Discussion

It is worth noting that (14) does not describe *non-zonal barotropic flows*. Indeed, if we split $P^{(0)}$ into barotropic and baroclinic components:

$$P^{(0)}(t, x, y, z) = P_{bt}^{(0)}(t, x, y) + P_{bc}^{(0)}(t, x, y, z), \qquad \int_{-1}^{0} P_{bc}^{(0)}(t, x, y, z)\,\mathrm{d}z = 0,$$

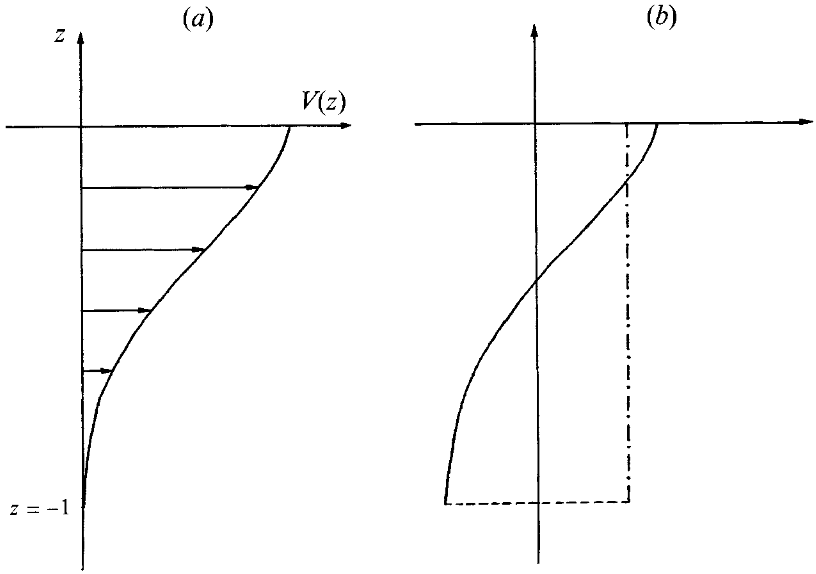

FIGURE 1. A meridional flow with strong barotropic component. System (14) is not applicable. (*a*) The velocity profile of the flow. (*b*) Solid line: the baroclinic component, dash-dotted line: the barotropic component.

(14*d*) yields

$$\frac{\partial}{\partial x} P^{(0)}_{bt}(t, x, y) = 0,$$

which obviously indicates that the barotropic component is, to leading order, zonal:

$$\frac{\text{non-zonal barotropic component}}{\text{baroclinic component}} \sim \frac{\epsilon P^{(1)}}{P^{(0)}} \sim \epsilon \ll 1.$$

This condition should be treated as a constraint imposed on the initial conditions allowed.

The physical meaning of this constraint is obvious: system (14) describes *large*-scale flows, where the barotropic mode is much faster than the baroclinic mode. Indeed, using the dispersion relations of the two modes:

$$\omega_{bt} = -\frac{\beta k_x}{k^2}, \quad \omega_{bc} = -\frac{\beta k_x}{k^2 + (R'_d)^{-2}},$$

and taking into account that $(kR'_d)^2 \sim Ro$ (see (6) with $k \sim L^{-1}$), we obtain

$$T_{bt}/T_{bc} \sim Ro \ll 1.$$

As a result, the 'slow' asymptotic system (14) does not describe fast barotropic waves, which are assumed to be 'instantly' radiated away. A similar restriction on the initial conditions allowed takes place when we use the quasi-geostrophic approximation to filter out gravity waves from the equations of fluid dynamics.

In order to understand, which initial conditions comprise a strong non-zonal barotropic component and therefore cannot be used with system (14), consider the vertical profile $V(z)$ of an initial distribution of the meridional velocity. If $V(z)$ corresponds to a unidirectional 'thick' flow (figure 1), the barotropic and baroclinic components are of the same order and system (14) is not applicable. Two important

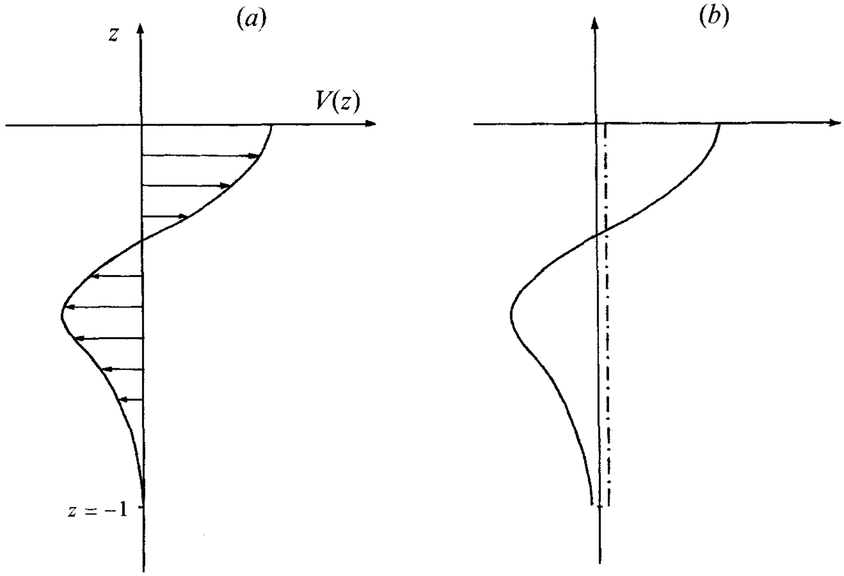

FIGURE 2. A meridional flow with counter-current. System (14) is applicable. (*a*) The velocity profile of the flow. (*b*) Solid line: the baroclinic component; dash-dotted line: the barotropic component.

examples of mainly baroclinic meridional flows can be seen in figures 2 and 3: the former includes a *counter-current*, the latter corresponds to a flow *localized near the surface of the ocean* – in both cases system (14) is applicable. Although in the latter case the baroclinic component dominates only in the upper 'active' layer (in the bottom layer the modes are equally weak), we may say that the baroclinic motion is stronger in the mean-square sense. Indeed, if h denotes the depth of the upper layer and z is the amplitude of the flow in it, the amplitudes of the modes in the upper and bottom layers are

$$V_{bc} \sim \begin{cases} a & \text{if} \quad z > -h, \\ -\dfrac{h}{1-h}a & \text{if} \quad z < -h; \end{cases}$$

$$V_{bt} \sim \frac{h}{1-h}a$$

(see figure 2). Thus,

$$\int_{-1}^{0} (V_{bc})^2 \, \mathrm{d}z \sim ha^2 + \frac{h^2}{1-h}a \sim ha^2,$$

$$\int_{-1}^{0} (V_{bt})^2 \, \mathrm{d}z \sim (ha)^2;$$

and since $h \ll 1$, it follows that the baroclinic component is stronger. We shall return to the localized flows at the end of §5.

It is convenient to rewrite (14) in terms of the barotropic and baroclinic components of $P^{(1)}$:

$$P^{(1)} = \Psi(t,x,y) + Q(t,x,y,z),$$

where

$$\int_{-1}^{0} Q \, \mathrm{d}z = 0. \tag{15a}$$

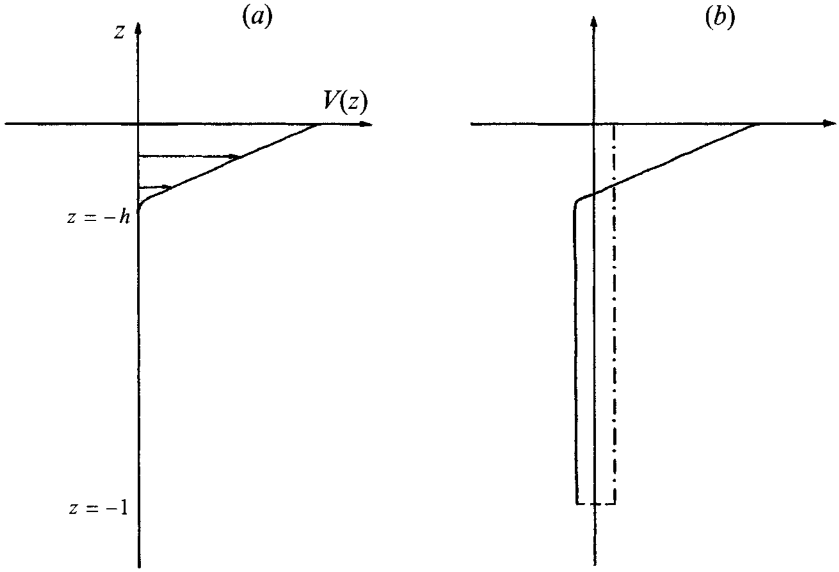

FIGURE 3. A meridional flow localized vertically. System (14) is applicable. (*a*) The velocity profile of the flow. (*b*) Solid line: the baroclinic component; dash-dotted line: the barotropic component.

Dropping the superscript$^{(0)}$, we write system (14) as follows:

$$\left.\begin{array}{c} w_z = \beta P_x, \\ w = 0 \quad \text{at} \quad z = -1; \end{array}\right\} \tag{15b}$$

$$J(P, P_z) = 0, \quad \int_{-1}^{0} P_x \, \mathrm{d}z = 0; \tag{15c}$$

$$P_{zt} + J(\Psi + Q, P_z) + J(P, Q_z) + w P_{zz} = 0, \tag{15d}$$

$$\beta \Psi_x + \int_{-1}^{0} J(P, \Delta P) \, \mathrm{d}z = 0. \tag{15e}$$

It should be recalled here that $w(t,x,y,z)$ is the vertical velocity, $P(t,x,y,z)$ describes the baroclinic and zonal barotropic components of the flow, $\Psi(t,x,y)$ is the amplitude of the barotropic non-zonal component and $Q(t,x,y,z)$ is the first correction to the baroclinic pressure field.

Although system (15) looks very complicated, it includes at least two tractable particular cases.

4. Stability of zonal flows with both vertical and horizontal shear

A steady zonal current is described by the following solution to system (15):

$$P = P(y,z), \quad w = 0, \quad \Psi = \Psi(y), \quad Q = 0. \tag{16}$$

In this section we shall examine the stability of (16) with respect to small-amplitude perturbations.

Linearizing (15) against the background of (16), we seek a harmonic-wave solution:

$$\begin{aligned} P(t,x,y,z) &= P(y,z) + p(y,z) \exp[\mathrm{i}k(ct - x)], \\ w(t,x,y,z) &= w(y,z) \exp[\mathrm{i}k(ct - x)], \\ Q(t,x,y,z) &= q(y,z) \exp[\mathrm{i}k(ct - x)], \\ \Psi(t,x,y) &= \Psi(y) + \psi(y) \exp[\mathrm{i}k(ct - x)]; \end{aligned}$$

where k and c are respectively the wavenumber and the phase speed of disturbances. Substitution of these equalities into (15) gives

$$\left.\begin{aligned} w_z &= -\mathrm{i}k\beta P, \\ w &= 0 \quad \text{at} \quad z=-1; \end{aligned}\right\} \tag{17a}$$

$$-pP_{zy}+P_y p_z = 0, \quad \int_{-1}^{0} p\,\mathrm{d}z = 0; \tag{17b}$$

$$\mathrm{i}k[cp_z-(\psi+q)P_{zy}+\Psi_y p_z+P_y q_z]+wP_{zz} = 0, \tag{17c}$$

$$\int_{-1}^{0} q\,\mathrm{d}z = 0, \tag{17d}$$

$$\beta\psi+\int_{-1}^{0}[pP_{yyy}+P_y(p_{yy}-k^2p)]\,\mathrm{d}z = 0. \tag{17e}$$

As this boundary-value problem is of first order with respect to z, the vertical structure of the solution can be found explicitly. Introducing

$$U(y,z) = -P_y,$$

we can express $p(y,z)$ and $w(y,z)$ in the form

$$p = A(y)\,U(y,z), \quad w = -\mathrm{i}k\beta A(y)\int_0^z U(y,z')\,\mathrm{d}z'; \tag{18}$$

where $A(y)$ is an undetermined function which describes the horizontal structure of the disturbance. Substitution of (18) into (17c, e) gives

$$Uq_z-U_z q = [(c+\Psi_y)A+\psi]\,U_z-\beta AP_{zz}\int_0^z U(y,z')\,\mathrm{d}z', \tag{19a}$$

$$\psi = [(FA_y)_y-k^2FA]; \tag{19b}$$

where

$$F = \frac{1}{\beta}\int_{-1}^{0} U^2\,\mathrm{d}z. \tag{19c}$$

Equation (19a) can be readily solved:

$$q = A(c+\Psi_y)+\psi+\beta AU(y,z)\int_{z_0}^{z}\frac{P_{z'z'}(y,z')}{U^2(y,z')}\int_{-1}^{z'}U(y,z'')\,\mathrm{d}z''\,\mathrm{d}z'+B(y)\,U, \tag{20}$$

where $B(y)$ is an undetermined function and $z_0(y)$ is such that $U[y,z_0(y)] \neq 0$. Obviously, $q(y,z)$ is continuous even at those points where $U=0$. Now we substitute (20) into (17d):

$$-(FA_y)_y+(c+k^2F+G)\,A = 0, \tag{21a}$$

where

$$G(y) = \Psi_y-\beta\int_{-1}^{0}U(y,z)\int_{z_0}^{z}\frac{P_{z'z'}(y,z')}{U^2(y,z')}\int_{-1}^{z'}U(y,z'')\,\mathrm{d}z''\,\mathrm{d}z'\,\mathrm{d}z$$

and $F(y)$ is given by (19c). Equation (21a) and the boundary condition

$$A\to 0 \quad \text{as} \quad y\to\pm\infty \tag{21b}$$

form an eigenvalue problem for $A(y)$ and c (the latter is the eigenvalue). Although one cannot solve (21) in general form (for unspecified $F(y)$ and $G(y)$), it is possible to prove that $\operatorname{Im} c = 0$ (a similar equation was considered by Benilov 1992). Multiplying (21 *a*) by A^* (the asterisk denotes complex conjugate) and integrating with respect to y over $(-\infty, \infty)$, we obtain

$$I_1 + I_2 c = 0,$$

where
$$I_1 = \frac{1}{\beta}\int_{-\infty}^{\infty} [F|A_y|^2 + (k^2F + G)|A|^2]\,\mathrm{d}y, \quad I_2 = \int_{-\infty}^{\infty} |A|^2\,\mathrm{d}y.$$

Since $I_2 \neq 0$, the phase speed $c = -I_1/I_2$ is real. The non-existence of unstable wave perturbations is proven, which is usually accepted as a proof of stability.

Discussion

It should be emphasized that the above proof of stability fails if $F(y)$ (given by (19 *c*)) vanishes at a finite value of y. Indeed, in this case the coefficient of the highest derivative in (21 *a*) vanishes, the eigenfunction has a singularity and the integrals I_1 and I_2 may diverge (this possibility was missed by in the two-layer study of Benilov 1992). However, this case is of a little practical interest, as it assumes that the level surface of zero velocity is ideally vertical (see (19 *c*)).

It is also worth noting that the stability of a flow within the framework of system (15) does not necessarily guarantee its stability within the framework of the original equations (7). Moreover, the results obtained in the two-layer case (Benilov 1992) indicate that zonal flows are weakly unstable with respect to short ($kR_d \sim 1$) disturbances even if the β-effect is strong. Nevertheless, this effect is unlikely to 'destroy' the flow: it may be conjectured that such short-wave instability leads to randomization of unstable disturbances, and the resulting turbulent friction stabilizes the flow. Eventually, the instability may saturate at some level.

In principle, a disturbance, whose length is comparable with the width of the flow, could destroy the latter, but the above results prove that all such disturbances are stable.

5. Separation of variables in system (15)

It was demonstrated by Benilov (1993) that the vertical and horizontal spatial variables in the equations which govern the case of weak β-effect can be separated. In this section we shall obtain a similar separable solution for the case of strong β-effect (i.e. for system (15)).

We shall seek a solution to (15) in the form

$$\left.\begin{aligned} P(t,x,y,z) &= \Phi(t,x,y)\,\phi_z(z), \\ w(t,x,y,z) &= \beta\Phi_x(t,x,y)\,\phi(z), \end{aligned}\right\} \tag{22}$$

where $\phi(z)$ describes the vertical profile of the baroclinic 'quasi-mode' and satisfies the boundary conditions

$$\phi = 0 \quad \text{at} \quad z = -1, 0. \tag{23a}$$

We shall also assume that

$$\int_{1}^{0} (\phi_z)^2\,\mathrm{d}z = 1. \tag{23b}$$

Substitution of (22) into (15 *e*) yields the equation for the barotropic-mode amplitude:

$$\beta\Psi_x + J(\Phi, \Delta\Phi) = 0. \tag{24}$$

Equation (15*c*), in turn, gives

$$J(Q,\Phi)\,\phi_{zz}+J(\Phi,Q_z)\,\phi_z = -[\Phi_t+J(\Psi,\Phi)]\,\phi_{zz}-\beta\Phi_x\,\Phi\phi\phi_{zzz}. \tag{25}$$

We seek a solution to (25) in the form

$$Q(t,x,y,z) = Q_1(t,x,y)+Q_2(t,x,y)f(z). \tag{26}$$

Substituting (26) into (25) and (15*d*), we get

$$\Phi_t+J(\Psi,\Phi)+J(Q_1,\Phi) = 0, \tag{27a}$$

$$\beta\Phi\Phi_x+J(Q_2,\Phi) = 0, \tag{27b}$$

$$f\phi_{zz}-f_z\,\phi_z = \phi\phi_{zz}, \tag{27c}$$

$$Q_1+\gamma Q_2 = 0; \tag{27d}$$

where

$$\gamma = \int_{-1}^{0} f(z)\,\mathrm{d}z \tag{28a}$$

characterizes the vertical density advection (the last term in (15*c*)). Equation (27*c*) can be readily solved:

$$f = \phi_z \int_{z_0}^{z} \frac{\phi(z')\,\phi_{z'z'z'}(z')}{[\phi_{z'}(z')]^2}\,\mathrm{d}z', \tag{28b}$$

where z_0 is an arbitrary point such that $\phi_z(z_0) \neq 0$. Multiplying (27*b*) by γ, adding it to (27*a*) and taking into account (27*d*), we obtain the following equation for the baroclinic-quasi-mode amplitude:

$$\Phi_t+J(\Psi,\Phi)+\gamma\beta\Phi\Phi_x = 0. \tag{29}$$

Equations (24) and (29) form a closed system for $\Psi(t,x,y)$ and $\Phi(t,x,y)$. Although these are *two*-dimensional functions, solution (22) describes *three*-dimensional flows.

In principle, the system (24), (29) can be generalized for an arbitrary number of baroclinic quasi-modes, each of which is defined in a separate layer. We shall not dwell on this question in detail, but refer the reader to Benilov (1993) where the 'n-quasi-mode' equations were derived for the (similar) case of weak β-effect.

Discussion

As mentioned above, similar separable solutions with arbitrary vertical profile ('quasi-modes') were found in the case of weak β-effect (Benilov 1993). The fact that they have arisen again indicates that quasi-modes might be an inherent feature of the asymptotic dynamics of all large-amplitude geostrophic flows.

Remarkably, the analysis of experimental data (e.g. Vasilenko & Mirabel 1977), as well as recent numerical results (Killworth 1992), suggest that in many instances the baroclinic component of oceanic flows can be approximated by a single baroclinic mode. This is a strong argument in favour of our model.

In conclusion, we shall calculate the coefficient of vertical density advection γ for the simplest specific case where the profile of the quasi-mode is linear in the upper layer of thickness h and constant in the bottom layer (see figure 2). Choosing the form of $\phi(z)$ to satisfy constraints (23), we get

$$\phi = \frac{1}{(\frac{1}{3}h^3-\frac{1}{4}h^4)^{\frac{1}{2}}}[(z+h)\,H(z+h)-\tfrac{1}{2}h^2], \tag{30}$$

where $H(z)$ is the Heaviside step function: $H(z \geqslant 0) = 1, H(z < 0) = 0$. Substitution of (30) into (28) gives

$$\gamma = -\frac{(1-h)^2}{(\frac{1}{3}h^3 - \frac{1}{4}h^4)^{\frac{1}{2}}}.$$

Evidently, if $h \rightarrow 0$, $\gamma \approx -h^{-\frac{3}{2}} \rightarrow \infty$, which indicates that the limit of a 'thin' upper layer is singular. Substituting (30) into (22) and writing the geostrophic criterion as $P \ll \epsilon^{-1}$, we obtain the condition $h \gg Ro$. Thus, in application to surface-localized flows, the system (24), (29) is valid only in the narrow asymptotic region

$$1 \gg h \gg Ro.$$

This agrees with the results obtained by Cushman-Roisin *et al.* (1992) for the two-layer stratification, where the limit $h \lesssim Ro$ yields an extra term in the baroclinic-mode equation.

6. Conclusions

The main results of this paper are:

(i) the derivation of the asymptotic system (15) which governs the dynamics of geostrophic flows with large displacement of isopycnal surfaces in the limit of strong β-effect;

(ii) the proof of stability of a zonal flow with both vertical and horizontal shear within the framework of system (15);

(iii) derivation of the two-dimensional system (24), (29) which describes a superposition of the barotropic mode and baroclinic 'quasi-mode', the vertical profile of the latter being arbitrary.

It should be emphasized that the results obtained are not applicable to flows with strong non-zonal barotropic component, or localized in a thin 'active' layer where the parameter

$$h \sim \frac{\text{thickness of the active layer}}{\text{total depth of the ocean}}$$

is of the order of, or less than, the Rossby number. As ocean data indicate that $h \sim \frac{1}{3} - \frac{1}{10}$, the latter case deserves an additional consideration.

It is also worth noting that the comparison of the two-layer model (Benilov 1992) and the continuous model of stratification (Benilov 1993 and the present paper) demonstrates a remarkable agreement: both models indicate stability/instability in the cases of strong/weak β-effect, respectively. This suggests that the two-layer model is a good approximation for all large-amplitude geostrophic flows.

REFERENCES

BENILOV, E. S. 1992 Large-amplitude geostrophic dynamics: the two-layer model. *Geophys. Astrophys. Fluid Dyn.* **66**, 67–79.

BENILOV, E. S. 1993 Baroclinic instability of large-amplitude geostrophic flows. *J. Fluid Mech.* **251**, 501–514.

CUSHMAN-ROISIN, B., SUTYRIN, G. G. & TANG, B. 1992 Two-layer geostrophic dynamics. Part 1: Governing equations. *J. Phys. Oceanogr.* **22**, 117–127.

KILLWORTH, P. D. 1992 An equivalent-barotropic mode in the Fine Resolution Antarctic Model. *J. Phys. Oceanogr.* **22**, 1379–1387.

KILLWORTH, P. D., PALDOR, N. & STERN, M. E. 1984 Wave propagation and growth on a surface front in a two-layer geostrophic current. *J. Mar. Res.* **42**, 761–785.

PALDOR, N. & GHIL, M. 1990 Finite-wavelength instabilities of a coupled density front. *J. Phys. Oceanogr.* **20**, 114–123.

RODEN, G. I. 1975 On North Pacific temperature, salinity, sound velocity and density fronts and their relation to the wind and energy flux fields. *J. Phys. Oceanogr.* **5**, 557–571.

SAKAI, S. 1989 Rossby–Kelvin instability: a new type of ageostrophic instability caused by a resonance between Rossby waves and gravity waves, *J. Fluid Mech.* **202**, 149–176.

VASILENKO, V. M. & MIRABEL, A. P. 1977 On the vertical structure of oceanic currents. *Acad. Sci. USSR, Izv. Atmos. Ocean. Phys.* **13** (3), 328–331.